1. Tail lights of cars have 6 W lamps operating at a voltage of 12 V.

(a) (6 Pts.) What is the resistance of the lamp?

(b) (6 Pts.) If 10 of these lamps are connected in series and this combination is connected to a 120 V voltage source, how much charge passes through the circuit in one hour?

(c) (7 Pts.) If 10 of these lamps are connected in parallel and this combination is to be connected to a 120 V voltage source, what is the value of the resistance which must be connected in series with the combination so that each lamp generates $6\ W$ of power?

(d) (6 Pts.) What will be the current supplied by the battery in part (c)?

Solution:

(a)

$$P = V I \quad \rightarrow \quad I = \frac{P}{V} = \frac{6\ \text{W}}{12\ \text{V}} = 0.5\ \text{A}, \qquad R = \frac{V}{I} = \frac{12\ \text{V}}{0.5\ \text{A}} \quad \rightarrow \quad R = 24\ \Omega.$$

(b) For 10 lamps connected in series, we have $R_{\text{eq}} = 10\ R = 240\ \Omega$.

$$I = \frac{V}{R_{\text{eq}}} = \frac{120\ V}{240\ \Omega} \quad \rightarrow \quad I = 0.5\ \text{A}, \qquad Q = I\,t = (0.5\ \text{A})(3600\ \text{s}) \quad \rightarrow \quad Q = 1800\ \text{C}.$$

(c) For 10 lamps connected in parallel, we have

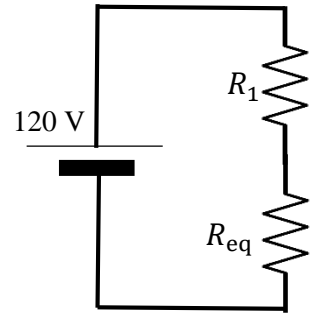$$\frac{1}{R_{\text{eq}}} = \frac{10}{R} \quad \rightarrow \quad R_{\text{eq}} = \frac{R}{10} = 2.4\ \Omega.$$

For each lamp to generate $6\ W$ of power we need to have $I = 0.5$ A. So the total current through the parallel combination of 10 lamps will be 5 A.

$$R_{\text{Tot}} = R_1 + R_{\text{eq}}, \qquad 120\ \text{V} = (5\ \text{A})R_{\text{Tot}} \quad \rightarrow \quad R_1 + R_{\text{eq}} = \frac{120\ V}{5\ A} = 24\ \Omega.$$

Hence $R_1 = 24\ \Omega - 2.4\ \Omega \quad \rightarrow \quad R_1 = 21.6\ \Omega$.

(d)

$$I = \frac{V}{R_{\text{Tot}}} = \frac{120\ V}{24\ \Omega} = 5\ \text{A}.$$

2. The switch S in the circuit shown has been open for a long time and the capacitor 2C is uncharged. At time $t = 0$ the switch S is closed.

(a) (5 Pts.) What is the charge on the capacitor C before the switch is closed?

(b) (5 Pts.) What is the current coming out of the battery before the switch is closed?

(c) (8 Pts.) Find the current I_2 on resistor R as a function of time for $t > 0$ after the switch is closed.

(d) (7 Pts.) What is the time constant of the circuit after the switch is closed?
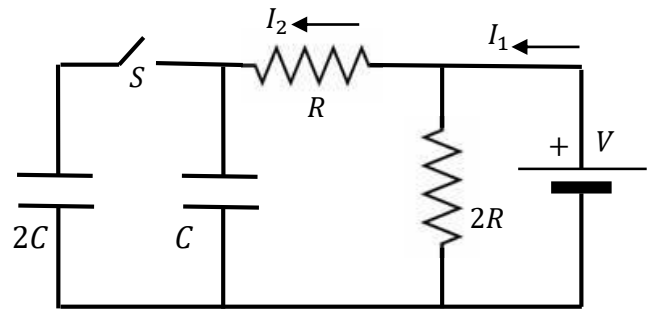
Solution: (a) At steady state before the switch S is closed we have $I_2 = 0$ and voltage on the capacitor C is equal to the voltage on the resistance $2R$, which is equal to V. Therefore, $Q_i = C\,V$.

(b) Before the switch S is closed $I_1 = V/2R$.

(c) At the time the switch is closed, equivalent circuit is as shown. Writing Kircchoff's loop rule, we have

$$V - I_2 R - V_{3C} = 0, \qquad I_2 = \frac{dQ}{dt}, \qquad V_{3C} = \frac{Q}{3C} \quad \rightarrow \quad \frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{3RC}.$$
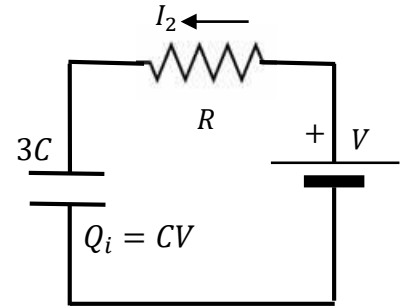
Writing this differential equation as

$$\frac{dQ}{\dfrac{Q}{3RC} - \dfrac{V}{R}} = -dt$$

and integrating with the initial condition that $Q(t = 0) = CV$, we get

$$Q(t) = 3CV\left(1 - \frac{2}{3}e^{-t/3RC}\right) \quad \rightarrow \quad I_2 = \frac{dQ}{dt} = \frac{2}{3}\frac{V}{R}e^{-t/3RC}.$$

(d)

$$\tau = 3RC.$$

3. A particle of mass M with no charge is moving with velocity $\vec{v} = v_0\,\hat{\imath}$, while another particle with the same mass but with charge $Q > 0$ is moving parallel to it at a distance D with a higher speed $v_1 > v_0$. Both particles enter a region of constant magnetic field simultaneously at $t = 0$, as shown in the figure. The magnitude of the magnetic field is B. We know that the two particles collide.

(a) (4 Pts.) Is the magnetic field into the page or out of the page? Why?

(b) (7 Pts.) When do they collide?

(c) (7 Pts.) Find v_0 in terms of v_1, B, Q, M and D.

(d) (7 Pts.) If the particles stick to each other and form a particle of mass $2M$ when they collide, what would be the period of their subsequent motion?
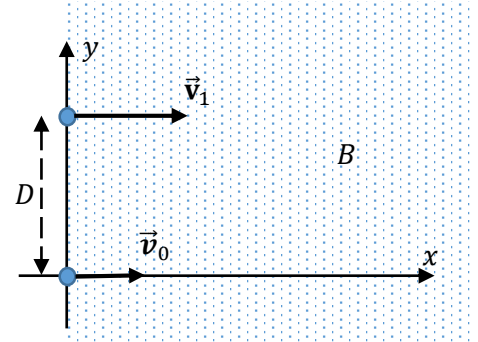
Solution: (a)

$$\vec{F} = Q\,\vec{v}_1 \times \vec{B}, \qquad \vec{v}_1 = v_1\,\hat{\imath}, \qquad \vec{F} = Qv_1B\,(-\hat{\jmath}) \text{ (in order to collide)}.$$

Therefore, we must have $\vec{B} = B\,\hat{k}$, out of the page.

(b) Trajectory of the charged particle will be a circle with radius R, such that

$$Qv_1B = \frac{Mv_1^2}{R} \quad \rightarrow \quad R = \frac{Mv_1}{QB}.$$
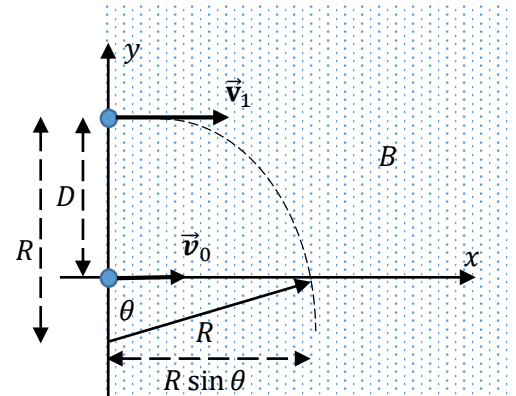
From the figure, we have

$$\cos\theta = \frac{R-D}{R} \quad \rightarrow \quad \theta = \arccos\left(1 - \frac{D}{R}\right).$$

On the other hand, time to complete one full revolution (period) is

$$T = \frac{2\pi R}{v_1} \quad \rightarrow \quad t = \frac{\theta}{2\pi}T = \frac{M}{QB}\arccos\left(1 - \frac{QBD}{Mv_1}\right)$$

is the time at which the two particles collide.

(c) During this time chargeless particle should cover the distance

$$R\sin\theta = \sqrt{R^2 - (R-D)^2}.$$

Hence,

$$\sqrt{R^2 - (R-D)^2} = v_0 t \quad \rightarrow \quad v_0 = \frac{\sqrt{\dfrac{2QBDv_1}{M} - \left(\dfrac{QBD}{M}\right)^2}}{\arccos\left(1 - \dfrac{QBD}{Mv_1}\right)}.$$

(d) For any particle of charge q and mass m in a magnetic field with magnitude B, we have $T = \frac{2\pi m}{qB}$. When the two particles stick together $q = Q$ and $m = 2M$. Therefore
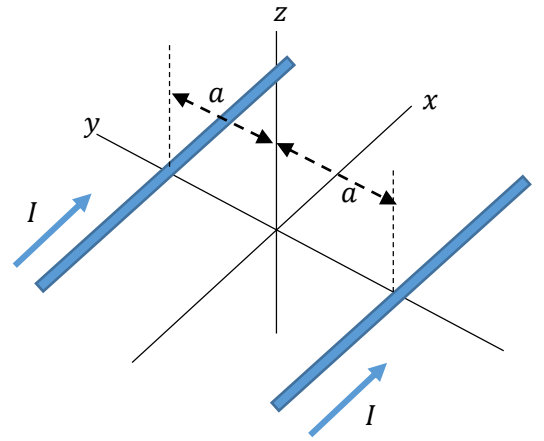
$$T = \frac{4\pi M}{QB}.$$

4. The figure shows two infinitely long straight wires situated at $y = \pm a, z = 0$ carrying parallel curents in the x-direction.

(a) (8 Pts.) What is the magnetic field (vector) on the y-axis between the two wires $-a < y < a$?

(b) (8 Pts.) What is the magnetic field (vector) on the z-axis?

(c) (9 Pts.) At what point on the positive z-axis is the magnetic field magnitude a maximum? What is this maximum value?

Solution:

(a)

$$\vec{B}(y) = \left(|\vec{B}_2| - |\vec{B}_1|\right) \hat{k}$$

Using Ampère's law, we have

$$B_1 = \frac{\mu_0 I}{2\pi(a-y)}, \qquad B_2 = \frac{\mu_0 I}{2\pi(a+y)}$$

Hence,

$$\vec{B}(y) = \frac{\mu_0 I}{2\pi}\left(\frac{1}{a+y} - \frac{1}{a-y}\right)\hat{k} = \frac{-\mu_0 I y\,\hat{k}}{\pi(a^2 - y^2)}.$$

(b)

From Ampère's law and symmetry

$$B_1 = B_2 = \frac{\mu_0 I}{2\pi\sqrt{a^2 + z^2}}.$$

$$\vec{B}(z) = 2B_1 \cos\theta\ (-\hat{j})$$

$$\cos\theta = \frac{z}{\sqrt{a^2 + z^2}}$$

$$\vec{B}(z) = \frac{\mu_0 I\,z}{\pi(a^2 + z^2)}\ (-\hat{j})$$

(c)

$$\frac{d}{dz}B(z) = \frac{\mu_0 I}{\pi}\left(\frac{a^2 + z^2 - 2z^2}{(a^2 + z^2)^2}\right) = 0 \quad \rightarrow \quad z = \pm a.$$

Maximum at $z = a$ on the positive z-axis.

$$B(a) = \frac{\mu_0 I}{2\pi a}.$$